

Table of Contents

9	*Continuous Mappings (General Theory)	1
9.1	Metric Spaces	1
9.1.1	Definition and Examples	1
9.1.2	Open and Closed Subsets of a Metric Space	5
9.1.3	Subspaces of a Metric space	7
9.1.4	The Direct Product of Metric Spaces	7
9.1.5	Problems and Exercises	8
9.2	Topological Spaces	9
9.2.1	Basic Definitions	9
9.2.2	Subspaces of a Topological Space	13
9.2.3	The Direct Product of Topological Spaces	13
9.2.4	Problems and Exercises	14
9.3	Compact Sets	15
9.3.1	Definition and General Properties of Compact Sets ...	15
9.3.2	Metric Compact Sets	16
9.3.3	Problems and Exercises	18
9.4	Connected Topological Spaces	19
9.4.1	Problems and Exercises	20
9.5	Complete Metric Spaces	21
9.5.1	Basic Definitions and Examples	21
9.5.2	The Completion of a Metric Space	24
9.5.3	Problems and Exercises	27
9.6	Continuous Mappings of Topological Spaces	28
9.6.1	The Limit of a Mapping	28
9.6.2	Continuous Mappings	30
9.6.3	Problems and Exercises	33
9.7	The Contraction Mapping Principle	34
9.7.1	Problems and Exercises	40
10	*Differential Calculus from a General Viewpoint	41
10.1	Normed Vector Spaces	41
10.1.1	Some Examples of Vector Spaces in Analysis	41
10.1.2	Norms in Vector Spaces	42
10.1.3	Inner Products in Vector Spaces	45

10.1.4	Problems and Exercises	48
10.2	Linear and Multilinear Transformations	49
10.2.1	Definitions and Examples	49
10.2.2	The Norm of a Transformation	51
10.2.3	The Space of Continuous Transformations	56
10.2.4	Problems and Exercises	60
10.3	The Differential of a Mapping	61
10.3.1	Mappings Differentiable at a Point	61
10.3.2	The General Rules for Differentiation	62
10.3.3	Some Examples	63
10.3.4	The Partial Derivatives of a Mapping	70
10.3.5	Problems and Exercises	71
10.4	The Finite-increment (Mean-value) Theorem	73
10.4.1	The Finite-increment Theorem	73
10.4.2	Some Applications of the Finite-increment Theorem	75
10.4.3	Problems and Exercises	79
10.5	Higher-order Derivatives	80
10.5.1	Definition of the n th Differential	80
10.5.2	Derivative with Respect to a Vector	81
10.5.3	Symmetry of the Higher-order Differentials	82
10.5.4	Some Remarks	84
10.5.5	Problems and Exercises	86
10.6	Taylor's Formula and the Study of Extrema	86
10.6.1	Taylor's Formula for Mappings	86
10.6.2	Methods of Studying Interior Extrema	87
10.6.3	Some Examples	89
10.6.4	Problems and Exercises	94
10.7	The General Implicit Function Theorem	96
10.7.1	Problems and Exercises	104
11	Multiple Integrals	107
11.1	The Riemann Integral over an n -Dimensional Interval	107
11.1.1	Definition of the Integral	107
11.1.2	The Lebesgue Criterion for Riemann Integrability	109
11.1.3	The Darboux Criterion	114
11.1.4	Problems and Exercises	116
11.2	The Integral over a Set	117
11.2.1	Admissible Sets	117
11.2.2	The Integral over a Set	118
11.2.3	The Measure (Volume) of an Admissible Set	119
11.2.4	Problems and Exercises	121
11.3	General Properties of the Integral	122
11.3.1	The Integral as a Linear Functional	122
11.3.2	Additivity of the Integral	122
11.3.3	Estimates for the Integral	123

11.3.4	Problems and Exercises	126
11.4	Reduction of a Multiple Integral to an Iterated Integral	127
11.4.1	Fubini's Theorem.....	127
11.4.2	Some Corollaries	129
11.4.3	Problems and Exercises	133
11.5	Change of Variable in a Multiple Integral	135
11.5.1	Statement of the Problem. Heuristic Considerations...	135
11.5.2	Measurable Sets and Smooth Mappings	136
11.5.3	The One-dimensional Case	138
11.5.4	The Case of an Elementary Diffeomorphism in \mathbb{R}^n	140
11.5.5	Composite Mappings and Change of Variable	142
11.5.6	Additivity of the Integral.....	142
11.5.7	Generalizations of the Change of Variable Formula ...	143
11.5.8	Problems and Exercises	147
11.6	Improper Multiple Integrals.....	150
11.6.1	Basic Definitions	150
11.6.2	The Comparison Test	153
11.6.3	Change of Variable in an Improper Integral	156
11.6.4	Problems and Exercises	158
12	Surfaces and Differential Forms in \mathbb{R}^n	161
12.1	Surfaces in \mathbb{R}^n	161
12.1.1	Problems and Exercises	170
12.2	Orientation of a Surface	170
12.2.1	Problems and Exercises	177
12.3	The Boundary of a Surface and its Orientation	178
12.3.1	Surfaces with Boundary	178
12.3.2	The Induced Orientation on the Boundary.....	181
12.3.3	Problems and Exercises	184
12.4	The Area of a Surface in Euclidean Space	185
12.4.1	Problems and Exercises	191
12.5	Elementary Facts about Differential Forms.....	195
12.5.1	Differential Forms: Definition and Examples	195
12.5.2	Coordinate Expression of a Differential Form	199
12.5.3	The Exterior Differential of a Form	201
12.5.4	Transformation under Mappings.....	204
12.5.5	Forms on Surfaces	207
12.5.6	Problems and Exercises	208
13	Line and Surface Integrals	211
13.1	The Integral of a Differential Form	211
13.1.1	The Original Problems. Examples	211
13.1.2	Integral over an Oriented Surface	218
13.1.3	Problems and Exercises	221
13.2	The Volume Element. Integrals of First and Second Kind	226

13.2.1	The Mass of a Lamina	226
13.2.2	The Area of a Surface as the Integral of a Form	227
13.2.3	The Volume Element	228
13.2.4	Cartesian Expression of the Volume Element	230
13.2.5	Integrals of First and Second Kind	231
13.2.6	Problems and Exercises	234
13.3	The Fundamental Integral Formulas of Analysis	237
13.3.1	Green's Theorem	237
13.3.2	The Gauss-Ostrogradskii Formula	242
13.3.3	Stokes' Formula in \mathbb{R}^3	245
13.3.4	The General Stokes Formula	248
13.3.5	Problems and Exercises	251
14	Elements of Vector Analysis and Field Theory	257
14.1	The Differential Operations of Vector Analysis	257
14.1.1	Scalar and Vector Fields	257
14.1.2	Vector Fields and Forms in \mathbb{R}^3	257
14.1.3	The Differential Operators grad, curl, div, and ∇	260
14.1.4	Some Differential Formulas of Vector Analysis	263
14.1.5	*Vector Operations in Curvilinear Coordinates	265
14.1.6	Problems and Exercises	274
14.2	The Integral Formulas of Field Theory	276
14.2.1	The Classical Integral Formulas in Vector Notation	276
14.2.2	The Physical Interpretation of div, curl, and grad	278
14.2.3	Other Integral Formulas	283
14.2.4	Problems and Exercises	285
14.3	Potential Fields	288
14.3.1	The Potential of a Vector Field	288
14.3.2	Necessary Condition for Existence of a Potential	289
14.3.3	Criterion for a Field to be Potential	290
14.3.4	Topological Structure of a Domain and Potentials	293
14.3.5	Vector Potential. Exact and Closed Forms	295
14.3.6	Problems and Exercises	298
14.4	Examples of Applications	302
14.4.1	The Heat Equation	302
14.4.2	The Equation of Continuity	304
14.4.3	Equations of Dynamics of Continuous Media	306
14.4.4	The Wave Equation	307
14.4.5	Problems and Exercises	309
15	*Integration of Differential Forms on Manifolds	313
15.1	A Brief Review of Linear Algebra	313
15.1.1	The Algebra of Forms	313
15.1.2	The Algebra of Skew-symmetric Forms	314
15.1.3	Linear Mappings and their Adjoints	317

15.1.4	Problems and Exercises	318
15.2	Manifolds	320
15.2.1	Definition of a Manifold	320
15.2.2	Smooth Manifolds and Smooth Mappings	325
15.2.3	Orientation of a Manifold and its Boundary	328
15.2.4	Partitions of Unity. Manifolds as Surfaces	331
15.2.5	Problems and Exercises	334
15.3	Differential Forms and Integration on Manifolds	337
15.3.1	The Tangent Space to a Manifold at a Point	337
15.3.2	Differential Forms on a Manifold	340
15.3.3	The Exterior Derivative	342
15.3.4	The Integral of a Form over a Manifold	343
15.3.5	Stokes' Formula	345
15.3.6	Problems and Exercises	347
15.4	Closed and Exact Forms on Manifolds	352
15.4.1	Poincaré's Theorem	352
15.4.2	Homology and Cohomology	356
15.4.3	Problems and Exercises	360
16	Uniform Convergence and Basic Operations of Analysis	363
16.1	Pointwise and Uniform Convergence	363
16.1.1	Pointwise Convergence	363
16.1.2	Statement of the Fundamental Problems	364
16.1.3	Convergence of a Family Depending on a Parameter	366
16.1.4	The Cauchy Criterion for Uniform Convergence	369
16.1.5	Problems and Exercises	371
16.2	Uniform Convergence of Series of Functions	372
16.2.1	Basic Definitions. Uniform Convergence of a Series	372
16.2.2	Weierstrass' M -test for Uniform Convergence	374
16.2.3	The Abel–Dirichlet Test	376
16.2.4	Problems and Exercises	380
16.3	Functional Properties of a Limit Function	381
16.3.1	Specifics of the Problem	381
16.3.2	Conditions for Two Limiting Passages to Commute	381
16.3.3	Continuity and Passage to the Limit	383
16.3.4	Integration and Passage to the Limit	386
16.3.5	Differentiation and Passage to the Limit	388
16.3.6	Problems and Exercises	393
16.4	*Subsets of the Space of Continuous Functions	397
16.4.1	The Arzelà–Ascoli Theorem	397
16.4.2	The Metric Space $C(K, Y)$	399
16.4.3	Stone's Theorem	400
16.4.4	Problems and Exercises	403

17 Integrals Depending on a Parameter	407
17.1 Proper Integrals Depending on a Parameter	407
17.1.1 The Basic Concept	407
17.1.2 Continuity of the Integral	408
17.1.3 Differentiation of the Integral	409
17.1.4 Integration of the Integral	413
17.1.5 Problems and Exercises	413
17.2 Improper Integrals Depending on a Parameter	415
17.2.1 Uniform Convergence with Respect to a Parameter ...	415
17.2.2 Continuity of an Integral Depending on a Parameter ..	423
17.2.3 Differentiation with Respect to a Parameter	426
17.2.4 Integration of an Improper Integral	429
17.2.5 Problems and Exercises	434
17.3 The Eulerian Integrals	437
17.3.1 The Beta Function	437
17.3.2 The Gamma Function	439
17.3.3 Connection Between the Beta and Gamma Functions .	442
17.3.4 Examples	443
17.3.5 Problems and Exercises	445
17.4 Convolution and Generalized Functions	449
17.4.1 Convolution in Physical Problems	449
17.4.2 General Properties of Convolution	451
17.4.3 Approximate Identities and Weierstrass' Theorem ...	454
17.4.4 *Elementary Concepts Involving Distributions	460
17.4.5 Problems and Exercises	471
17.5 Multiple Integrals Depending on a Parameter	476
17.5.1 Proper Multiple Integrals Depending on a Parameter .	476
17.5.2 Improper Multiple Integrals	477
17.5.3 Improper Integrals with a Variable Singularity	478
17.5.4 *Convolution in the Multidimensional Case	483
17.5.5 Problems and Exercises	493
18 Fourier Series and the Fourier Transform	499
18.1 Basic General Concepts Connected with Fourier Series	499
18.1.1 Orthogonal Systems of Functions	499
18.1.2 Fourier Coefficients and Fourier Series	506
18.1.3 *A Source of Orthogonal Systems	516
18.1.4 Problems and Exercises	520
18.2 Trigonometric Fourier Series	526
18.2.1 Basic Types of Convergence of Fourier Series	526
18.2.2 Pointwise Convergence of a Fourier Series	531
18.2.3 Smoothness and Decrease of Fourier Coefficients	540
18.2.4 Completeness of the Trigonometric System	545
18.2.5 Problems and Exercises	552
18.3 The Fourier Transform	560

18.3.1	Fourier Integral Representation	560
18.3.2	Differential Properties and the Fourier Transform.....	573
18.3.3	Main Structural Properties	576
18.3.4	Examples of Applications.....	581
18.3.5	Problems and Exercises	587
19	Asymptotic Expansions	595
19.1	Asymptotic Formulas and Asymptotic Series	597
19.1.1	Basic Definitions	597
19.1.2	General Facts about Asymptotic Series.....	602
19.1.3	Asymptotic Power Series	606
19.1.4	Problems and Exercises	609
19.2	The Asymptotics of Integrals (Laplace's Method)	612
19.2.1	The Idea of Laplace's Method.....	612
19.2.2	The Localization Principle for a Laplace Integral	615
19.2.3	Canonical Integrals and their Asymptotics.....	617
19.2.4	Asymptotics of a Laplace Integral	621
19.2.5	*Asymptotic Expansions of Laplace Integrals	624
19.2.6	Problems and Exercises	635
	Topics and Questions for Midterm Examinations	643
1.	Series and Integrals Depending on a Parameter	643
2.	Problems Recommended as Midterm Questions	644
3.	Integral Calculus (Several Variables)	646
4.	Problems Recommended for Studying the Midterm Topics	647
	Examination Topics	649
1.	Series and Integrals Depending on a Parameter	649
2.	Integral Calculus (Several Variables)	651
	References	653
1.	Classic Works	653
1.1	Primary Sources	653
1.2.	Major Comprehensive Expository Works.....	653
1.3.	Classical courses of analysis from the first half of the twentieth century.....	653
2.	Textbooks	654
3.	Classroom Materials	654
4.	Further Reading	655
	Index of Basic Notation	657
	Subject Index	661
	Name Index	679